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THE ENDOMORPHISM SEMIGROUP OF A SEMIGROUP AND ITS APPLICATION

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An endomorphism of a semigroup is one of important tools in the study of semigroups. In this paper , we represent the endomorphism semigroup by means of a wreath product and the greatest semilattice homomorphic image. And we apply to a left inverse extension of certain left regular band by a semilattice of groups.

1. The endomorphism semigroup of semigroups.

Let S be a semigroup. The set End(S) of all endomorphisms of S is a semigroup under the composition. The set Aut(S) of all automorphisms of S is a subgroup of End(S). End(S) [Aut(S)] is called <u>the endomorphism semigroup</u> [<u>the automorphism group</u>] <u>of</u> S. Now, let $S=\bigcup\{S_\alpha : \alpha\in\Gamma\}$ be the greatest semilattice decomposition of S. Let $H(S)=\bigcup\{Hom(S_\alpha,S_\beta): \alpha,\beta\in\Gamma\}\cup\{\square\}$ be a semigroup under composition , where $Hom(S_\alpha,S_\beta)$ is the set of all homomorphisms of S_α to S_β and $\square$ is the empty mapping. Consider the set $H(S)$ wr $End(\Gamma) = \{(k,\theta): k:\Gamma\longrightarrow H(S), \theta\in End(\Gamma)\}$ and define a multiplcation in $H(S)$ wr $End(\Gamma)$ as follows:

$$(k,\theta)(k',\theta') = (k\cdot\theta k',\theta\theta') ,$$

where $\gamma k = k_\gamma$ and $(k\cdot\theta k') = k_\gamma k'_{\gamma\theta}$ for all $\gamma\in\Gamma$. Then $H(S)$ wr $End(\Gamma)$ is a semigroup and we call it <u>the wreath product of</u> $H(S)$ <u>and</u> $End(\Gamma)$.

For each $\phi\in End(S)$, we define mappings $k:\Gamma\longrightarrow H(S)$ and $\theta: \Gamma\longrightarrow\Gamma$ as follows:

$$k_\alpha = \phi|_{S_\alpha} ,\quad S_\alpha\phi\subseteq S_{\alpha\theta} \quad \text{for all } \alpha\in\Gamma .$$

Then a mapping $f:\phi\longmapsto(k,\theta)$ is an isomorphism of End(S) into $H(S)$ wr $End(\Gamma)$. Thus we have the following lemma.

Lemma 1. $\quad End(S) \cong_{in} H(S) \text{ wr } End(\Gamma)$.

Next, we shall consider the condition that $(k,\theta) \in H(S)\text{wr}End(\Gamma)$ is contained in the image $Im(f)$ of f.

Lemma 2. Let $(k,\theta) \in H(S) \text{ wr } End(\Gamma)$. Then $(k,\theta) \in Im(f)$ if and only if

(i) $k_\alpha \in Hom(S_\alpha, S_{\alpha\theta})$ for all $\alpha \in \Gamma$,

(ii) $(xy)\cdot k_{\alpha\beta} = [x\cdot k_\alpha][y\cdot k_\beta]$ for all $x \in S_\alpha$, $y \in S_\beta$.

Now, we shall call this condition an H-property and let $H(S) \widetilde{\text{wr}} End(\Gamma) = \{(k,\theta) \in H(S) \text{ wr } End(\Gamma) : (k,\theta) \text{ has an H-property}\}$. By above lemmas we obtain the following theorem.

Theorem 3. Let $S = \bigcup\{S_\alpha : \alpha \in \Gamma\}$ be the greatest semilattice decomposition of a semigroup S. Then

$$End(S) \cong H(S) \widetilde{\text{wr}} End(\Gamma).$$

From now on, we identity elements of End(S) with elements of $H(S) \widetilde{\text{wr}} End(\Gamma)$.

Remark 1. Since $H(S) \text{ wr} \mathcal{T}_\Gamma$ is isomorphic with the semigroup of all row monomial $\Gamma \times \Gamma$-matrices over H(S), End(S) can represent by row monomial matrices.

2. Let $S = \bigcup\{S_\alpha : \alpha \in \Gamma\}$ be a strong semilattice of semigroups S_α determined by a transtive system $\{\phi_{\alpha,\beta} : \alpha \geq \beta\}$ of homomorphisms. If each S_α satisfy certain conditions (e.g. , S_α has an identity and no other idempotent or S_α is weakly cancellative), then an H-property is equivalent to that the following diagram is commutative:

$$\begin{array}{ccc} S_\alpha & \xrightarrow{k_\alpha} & S_{\alpha\theta} \\ {\scriptstyle \phi_{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle \phi_{\alpha\theta,\beta\theta}} \\ S_\beta & \xrightarrow{k_\beta} & S_{\beta\theta} \end{array} \qquad (\alpha \geq \beta,\ \alpha, \beta \in \Gamma)$$

(c.f., Petrich[2]).

2. The endomorphism semigroup of a successively annihilating chain of semigroups.

To each α in a chain Y assign a pairwise disjoint semigroup S_α, and on $S = \bigcup\{S_\alpha : \alpha \in Y\}$ define a multiplication $\circ$ by

$$a\circ b = b\circ a = b \quad \text{if } a \in S_\alpha,\ b \in S_\beta,\ \alpha > \beta,$$
$$a\circ b = ab \quad \text{if } a,b \in S_\alpha .$$

Then an semigroup $S(\circ)$ is called <u>a successively annihilating chain</u> Y <u>of semigroups</u> S_α (c.f., Petrich [2]).

We shall first consider an endomorphism of a successively annihilating chain of completely simple semigroups.

Let Y be a chain. To each $\alpha \in Y$ assign a pairwise disjoint completely simple semigroup $S_\alpha = M(G_\alpha : I_\alpha, \Lambda_\alpha : P_\alpha)$ where $P_\alpha = (p^{(\alpha)}_{\lambda,i})$ has been normalized. Let $S = \bigcup\{S_\alpha : \alpha \in Y\}$ be a successively annihilating chain of S_α.

Theorem 4. Let θ be an endomorphism of Y(i.e.,an isotone transformation of Y). We define a mapping $k : Y \longrightarrow H(S)$ as follows:

(i) The case $|\alpha\theta\theta^{-1}| > 1$.

Let i_o and λ_o be fixed elements of $I_{\alpha\theta}$ and $\Lambda_{\alpha\theta}$, respectively. Let β be an element of Y such that $\beta\theta = \alpha\theta$. Let ω be a homomorphism of G_β in $G_{\alpha\theta}$ such that $p^{(\beta)}_{\lambda,i}\omega = e_{\alpha\theta}$, the identity element of $G_{\alpha\theta}$, for all $i \in I_\beta$ and $\lambda \in \Lambda_\beta$. Define $k_\beta : S_\beta \longrightarrow S_{\alpha\theta}$ by, for all $i_\beta \in I_\beta$, $\lambda_\beta \in \Lambda_\beta$ and $g_\beta \in G_\beta$,

$$(i_\beta, g_\beta, \lambda_\beta)\cdot k_\beta = \begin{cases} (i_o, (g_\beta\omega)(p^{(\alpha\theta)}_{\lambda_o,i_o})^{-1}, \lambda_o) & \text{if } \beta \text{ is the least element in } \alpha\theta\theta^{-1}, \\ (i_o, (p^{(\alpha\theta)}_{\lambda_o,i_o})^{-1}, \lambda_o) & \text{otherwise.} \end{cases}$$

(ii) Other case (i.e., $|\alpha\theta\theta^{-1}| = 1$).

Define k_α by any homomorphism of S_α in $S_{\alpha\theta}$ (c.f.,[1]).

Then $(k,\theta) \in End(S) = H(S)\ \widehat{wr}\ End(Y)$ and conversely every endomorphism of S can be obtained in this fashion.

We shall use the following corollary in next section.

Corollary 5. Let $L = \bigcup\{L_\alpha : \alpha \in Y\}$ be a successively annihilating chain Y of left zero semigroups L_α. Let θ be an endomorphism of Y. We define a mapping $k: Y \longrightarrow H(L)$ as follows:

(i) Case $|\alpha\theta\theta^{-1}| > 1$.

Let l_o be a fixed element of $L_{\alpha\theta}$. Let β be an element of Y such that $\beta\theta = \alpha\theta$. Define $k_\beta : L_\beta \longrightarrow L_{\alpha\theta}$ by

$$xk_\beta = l_o \quad \text{for all } x \in L_\beta .$$

(ii) Case $|\alpha\theta\theta^{-1}| = 1$.

Define k_α by any mapping of L_α in $L_{\alpha\theta}$.

Then $(k,\theta) \in End(L) = H(L) \ \widehat{wr}\ End(Y)$ and conversely every endomorphism of L can be constructed in this manner.

Under some condition we next show that the endomorphism semigroup is a direct product of endomorphism semigroups of subsemigroups.

Let $\Gamma = \bigcup\{\Gamma_i : i \in Y\}$ be a semilattice which is a chain Y of semilattice Γ_i. Let $\{T_i : i \in Y\}$ be a family of semilattice indecomposable semigroups T_i such that $Hom(T_i, T_j) = \Box$ $[T_i \not\cong T_j]$ for all $i \neq j \in Y$. Let $S = \bigcup\{S_\alpha : \alpha \in \Gamma\}$ be a semigroup such that Γ is the structure semilattice of S and $S_\alpha \cong T_i$ if $\alpha \in \Gamma_i$. In Proposition 6 and Corollary 7, let $S = \bigcup\{S_i : i \in Y\}$ be a chain of semigroups S_i where $S_i = \bigcup\{S_\alpha : \alpha \in \Gamma_i\}$, and assume that T_i's satisfy conditions in the brackets for a statement on an automorphism group .

Proposition 6. $End(S) \cong {}_{in}\prod_{i \in Y} End(S_i)$.

$$[Aut(S) \cong {}_{in}\prod_{i \in Y} Aut(S_i)].$$

Corollary 7. Moreover, let S be a successively annihilating chain Y of semigroups S_i. Then

$$\mathrm{End}(S) \cong \prod_{i \in Y} \mathrm{End}(S_i).$$
$$[\mathrm{Aut}(S) \cong \prod_{i \in Y} \mathrm{Aut}(S_i)].$$

By the above result we need to consider the endomorphism semigroup as follows. Let S be a semigroup such that $S = \bigcup\{S_\alpha : \alpha \in \Gamma\}$ and $\phi_\alpha : S_\alpha \cong S_0$ for all $\alpha \in \Gamma$, where S_0 is a semilattice indecomposable semigroup.

Let k be a mapping of Γ into $\mathrm{End}(S_0)$ and let $\theta \in \mathrm{End}(\Gamma)$. Then we say that (k,θ) <u>has an H-property</u> if (k,θ) satisfy the following condition:

$$[a\cdot(\phi_\alpha k_\alpha \phi_{\alpha\theta}^{-1})][b\cdot(\phi_\beta k_\beta \phi_{\beta\theta}^{-1})] = (ab)\cdot(\phi_{\alpha\beta} k_{\alpha\beta} \phi_{(\alpha\beta)\theta}^{-1})$$
for all $a \in S_\alpha$, $b \in S_\beta$,

and $\mathrm{End}(S_0)\ \overline{\mathrm{wr}}\ \mathrm{End}(\Gamma) = \{(k,\theta) \in \mathrm{End}(S_0)\ \mathrm{wr}\ \mathrm{End}(\Gamma) : (k,\theta)$ has an H-property$\}$.

Proposition 8. Let S be the above semigroup. Then

$$\mathrm{End}(S) \cong \mathrm{End}(S_0)\ \overline{\mathrm{wr}}\ \mathrm{End}(\Gamma) \text{ and}$$
$$\mathrm{Aut}(S) \cong \mathrm{Aut}(S_0)\ \overline{\mathrm{wr}}\ \mathrm{Aut}(\Gamma).$$

Corollary 9. Let S_0 be a weakly cancellative semigroup or S_0 has an identity and no other idempotent. Then

$$\mathrm{End}(S_0 \times \Gamma) \cong \mathrm{End}(S_0) \times \mathrm{End}(\Gamma) \text{ and}$$
$$\mathrm{Aut}(S_0 \times \Gamma) \cong \mathrm{Aut}(S_0) \times \mathrm{Aut}(\Gamma).$$

3. An application to left inverse semigroup

A regular semigroup S is called <u>a left inverse semigroup</u> if the set E of all idempotents in S satisfy the identity xyx = xy. A structure theorem for left inverse semigroups have been given by M. Yamada [3] as follows:

<u>Structure Theorem</u>. Let Ω be an inverse semigroup and Y its

basic semilattice. Let $L=\bigcup\{L_\alpha:\ \alpha\in Y\}$ be a left legular band (i.e., L is a semilattice Y of left zero semigroups L_α). Let φ be a mapping of Ω into End(L) such that the family $\{\varphi_\omega:\ \omega\in\Omega\}$, $\varphi_\omega=\varphi(\omega)$, satisfies the following (C_1) and (C_2):

(C_1) Each φ_ω is an endomorphism on L such that $\varphi_\omega(L_\alpha)\subseteq L_{\omega\alpha(\omega\alpha)^{-1}}$ for all $\alpha\in Y$. In particular, for $\tau\in Y$, φ_τ is an inner endomorphism on L.

(C_2) $\delta_e\delta_f\varphi_\alpha\varphi_\beta=\delta_e\delta_f\varphi_{\alpha\beta}$ for $e\in L_{\alpha\alpha^{-1}}$, $f\in L_{(\alpha\beta)(\alpha\beta)^{-1}}$, $\alpha,\beta\in\Omega$ (where δ_e is the inner endomorphism on L induced by e).

Consider the set $L\underset{\varphi}{\otimes}\Omega=\{(e,\omega):\omega\in\Omega,\ e\in L_{\omega\omega^{-1}}\}$ and define multiplication in $L\underset{\varphi}{\otimes}\Omega$ as follows:

$$(e,\omega)(f,\tau)=(e\varphi_\omega(f),\omega\tau)\ .$$

Then $L\underset{\varphi}{\otimes}\Omega$ is a left inverse semigroup and conversely every left inverse semigroup can be constructed in this manner.

We shall call this $L\underset{\varphi}{\otimes}\Omega$ _the left inverse extension of_ L _by_ Ω and a mapping $\varphi:\Omega\longrightarrow\mathrm{End}(L)$ which satisfies (C_1) and (C_2) is called _a factor system of_ Ω _into_ L and we shall denote the set of all factor systems of Ω into L by $\Phi(\Omega,L)$. We shall define a relation $\sim$ on $\Phi(\Omega,L)$ as follows; let $\varphi_1,\varphi_2\in\Phi(\Omega,L)$,

$$\varphi_1\sim\varphi_2 \text{ if and only if } L\underset{\varphi_1}{\otimes}\Omega\cong L\underset{\varphi_2}{\otimes}\Omega\ .$$

The purpose of this section is to investigate $\Phi(\Omega,L)/\!\sim$ in case Ω is a chain of groups and L is a successively annihilating chain of left zero semigroups .

Let $L=\bigcup\{L_\alpha:\ \alpha\in Y\}$ be a successively annihilating chain Y of left zero semigroups L_α. Let $\Omega=\bigcup\{G_\alpha:\ \alpha\in Y\}$ be a chain Y of groups G_α determined by a transitive system $\{f_{\beta,\alpha}:\ \beta\leq\alpha\}$ of a homomorphism $f_{\beta,\alpha}:G_\alpha\longrightarrow G_\beta$.

First of all, we shall investigate a factor system of Ω in

L.

Theorem 10. Let $k = (k_{\beta,\alpha})$ be an element of $\prod_{\beta<\alpha} \mathrm{Hom}(G_\alpha, S(L_\beta))$ such that if $\gamma < \beta < \alpha$ then $k_{\gamma,\beta} f_{\beta,\alpha} = k_{\gamma,\alpha}$ where $S(L_\beta)$ is the symmetric group on L_β. Let ϕ be a mapping of Ω in L such that $\phi(G_\alpha) \subseteq L_\alpha$ for all $\alpha \in Y$ where if α is the greatest element in Y then assume that $\phi(G_\alpha) \subseteq \mathcal{J}(L_\alpha)$, the full transformation semigroup on L_α and $\phi(e_\alpha)$ is a constant mapping (e_α is an identity of G_α). Define $\varphi : \Omega \longrightarrow \mathrm{End}(L)$ by

$$x \cdot \varphi(g_\alpha) = \begin{cases} x \cdot k_{\beta,\alpha}(g_\alpha) & \text{if } \beta < \alpha, \\ \phi(g_\alpha) & \text{if } \beta \geqq \alpha,\ \alpha \text{ is not greatest}, \\ x \cdot \phi(g_\alpha) & \text{if } \beta = \alpha,\ \alpha \text{ is greatest}, \end{cases}$$

for all $x \in L_\beta$, $g_\alpha \in G_\alpha$, $\alpha, \beta \in \Omega$.

Then φ is a factor system of Ω in L. Conversely any factor system of Ω in L can be obtained in this fashion.

Hence we shall denote a factor system φ by $\varphi = ((k, \phi))$.

Theorem 11. Let $((k^{(1)}, \phi^{(1)}))$ and $((k^{(2)}, \phi^{(2)}))$ be factor systems of Ω in L. Then $((k^{(1)}, \phi^{(1)})) \sim ((k^{(2)}, \phi^{(2)}))$ if and only if there are an automorphism $(\psi, \theta) \in \mathrm{Aut}(\Omega)$ and bijections $p_\beta : L_\beta \longrightarrow L_{\beta\theta}$ ($\beta \in Y$) such that $k^{(1)}_{\beta,\alpha} = \rho_{p_\beta} k^{(2)}_{\beta\theta,\alpha\theta} \psi_\alpha$ for any $\alpha, \beta \in Y$ such that $\beta < \alpha$, where $\rho_{p_\beta} : S(L_{\beta\theta}) \longrightarrow S(L_\beta)$ is defined by $\rho_{p_\beta}(p) = p_\beta^{-1} p p_\beta$.

Corollary 12. Let $\Omega = G \times Y$ be a direct product of a group G and a well-ordered chain Y. Let $L = \bigcup \{L_\alpha : \alpha \in Y\}$ be a successively annihilating chain Y of left zero semigroups L_α.
On the set $\prod_{\beta \in Y^*} \mathrm{Hom}(G, S(L_\beta))$, define a relation $\approx$ by,
$k^{(1)} \approx k^{(2)}$ if and only if there are an automorphism $\psi \in \mathrm{Aut}(G)$ and $p_\beta \in S(L_\beta)$ ($\beta \in Y^*$) such that $k^{(1)}_\beta = \rho_{p_\beta} k^{(2)}_\beta \psi$ for all $\beta \in Y^*$

where $Y^{*} = Y \setminus \alpha_1$, α_1 is the greatest element of Y. Then there is a one-to-one correspondence between the classes of isomorphic left inverse extensions of L by Ω and the elements of $\prod_{\beta \in Y^{*}} \mathrm{Hom}(G, S(L_\beta))/\approx$.

Example. Let G be a cyclic group of order 2, $Y = \{0,1\}$ and $L = L_0 \cup L_1$. If $|L_0| = 2n$, then $|\Phi(\Omega, L)/\sim| = n+1$. If $|L_0| = \infty$ then $|\Phi(\Omega, L)/\sim| = \infty$.

Corollary 13. Let Y be a dense chain(i.e., if $\alpha > \beta$, there is an element γ such that $\alpha > \gamma > \beta$). Let Ω and L be successively annihilating chains Y of groups G_α and left zero semigroups L_α, respectively. Then there exist the only one left inverse extension of L by Ω , up to isomorphism.

References

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